Tutorial on Sensitivity Testing in Live Fire Test and Evaluation

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About This Publication

A sensitivity experiment is a special type of experimental design that is used when the response variable is binary and the covariate is continuous. Armor protection and projectile lethality tests often use sensitivity experiments to characterize a projectile’s probability of penetrating the armor. In this mini-tutorial we illustrate the challenge of modeling a binary response with a limited sample size, and show how sensitivity experiments can mitigate this problem. We review eight different single covariate sensitivity experiments and present a comparison of these designs using simulation. Additionally, we cover sensitivity experiments for cases that include more than one covariate, and highlight recent research in this area.

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Tutorial on Sensitivity Testing in Live Fire Test and Evaluation

Thomas Johnson
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Ray Chen
Executive Summary

A sensitivity experiment is a special type of sequential experimental design that is used for binary outcomes. In this tutorial we look at a common live fire test outcome — whether armor is penetrated or not by a projectile. Armor protection and projectile lethality tests often use sensitivity experiments to characterize a projectile’s probability of penetrating armor as a function of the projectile’s velocity. These tests are referred to as “sequential” because the experimental design is sequentially updated after each shot is recorded. Simply put, after every shot the velocity of the next projectile shot is updated based on previous test outcomes. Sensitivity experiments are often used in armor characterization testing when the objective is to estimate the velocity at which the projectile has a 50 percent probability of penetration. In past work, the authors compared numerous single factor sequential designs and concluded that 3Pod was best in terms of robustness to model misspecification, and accuracy.

Multi-factor sequential design, as the name suggests, deals with more than one continuous factor. Velocity is typically a primary factor for armor tests, but secondary factors include obliquity angle, yaw angle, armor temperature, and other physics-based continuous parameters that affect projectile penetration. Sequential design, and multi-factor sequential design in particular, are well-suited for Live Fire Test and Evaluation because such tests are often conducted in a controlled laboratory environments where precise control of multiple continuous factors is possible.

In this mini-tutorial we illustrate the challenge of modeling a binary response with a limited sample size, and show how sensitivity experiments can mitigate this problem. We review eight different single factor sensitivity experiments, and present a comparison of these designs using simulation. Additionally, we present sensitivity experiments for cases that include more than one factor, and highlight recent research in this area.
Tutorial on Sensitivity Experiments in Live Fire Test and Evaluation

Tom Johnson
Laura Freeman
Ray Chen
Sensitivity Experiments Best Practices
Outline

1. Introduction to Binary Response Experiments
2. Binary Response Test Design Challenges
3. 1-D Sensitivity Test Designs
4. 2-D Sensitivity Test Designs
5. Case Study: Greg Hutto
Introduction to Binary Response Experiments
Types of Binary Response Experiments

Pharmaceutical Industry
   Lethal dose
   Effective dose

Defense Industry
   Lethality of munitions
   Survivability of systems
   Armor Characterization
Defense Industry Requirements

“Munition shall have a V50 less than 2,000 ft/s”

“Armor shall have a v50 greater than 2,300 ft/s”

Historically, an arithmetic mean estimator is used to calculated V50
## Regression Models

<table>
<thead>
<tr>
<th>Link Name (distribution)</th>
<th>Probability of perforation $\pi(v)$</th>
<th>Velocity where probability of perforation is $\pi$, $\hat{V}_\pi$</th>
<th>Estimated $V_{50}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Logit (Logistic)</td>
<td>$\frac{e^{\hat{\beta}_0 + \hat{\beta}_1 v}}{1 + e^{\hat{\beta}_0 + \hat{\beta}_1 v}}$</td>
<td>$\frac{\ln\left(\frac{\pi}{1 - \pi}\right) - \hat{\beta}_0}{\hat{\beta}_1}$</td>
<td>$-\frac{\hat{\beta}_0}{\hat{\beta}_1}$</td>
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<tr>
<td>Probit (Normal)</td>
<td>$\Phi\left(\hat{\beta}_0 + \hat{\beta}_1 v\right)$</td>
<td>$\frac{\Phi^{-1}(\pi) - \hat{\beta}_0}{\hat{\beta}_1}$</td>
<td>$-\frac{\hat{\beta}_0}{\hat{\beta}_1}$</td>
</tr>
</tbody>
</table>

---

![Graph showing probability of penetration vs. velocity](image)
Binary Response Test Design Challenges
Binary Response Designs Need Special Consideration

<table>
<thead>
<tr>
<th>Run #</th>
<th>Velocity</th>
<th>Response</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>1500</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1500</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1500</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
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<td>0</td>
</tr>
<tr>
<td>5</td>
<td>3000</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>3000</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>3000</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>3000</td>
<td>1</td>
</tr>
</tbody>
</table>

“Evidence of perfect fit” yields bad logistic model fit
Binary Response Designs Need Special Consideration

<table>
<thead>
<tr>
<th>Run #</th>
<th>Velocity</th>
<th>Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1500</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1500</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1500</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1875</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>2625</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>3000</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>3000</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>3000</td>
<td>1</td>
</tr>
</tbody>
</table>

A zone of mixed results provides a good rough estimate of the logistic model curve
Zone of Mixed Results

Example of Separation
- Penetration
- No Penetration

Example of Quasi-Separation
- Penetration
- No Penetration

Example of No Separation (Zone of Mixed Results Exists)
- Penetration
- No Penetration
Test Designs to Achieve a Zone of Mixed Results

Sequential Methods with Initials Designs

Bayesian Methods
1-D Sensitivity Test Designs
# Up and Down

## Details of Implementation

### Rules
- If projectile **does** penetrate armor, decrease velocity.
- If projectile **does not** penetrate armor, increase velocity.

### Inputs
- Step size
- Velocity of projectile for trial number one

### Other details
- fixed step size
- step size calculated from anticipated standard deviation
- Initial shot typically taken at predicted V50

## Background
- Most well-known sequential experimentation procedure, primarily due to its ease of implementation
- Developed by Dixon in 1948

## Advantages
- Useful for estimating V50
- The rules are simple and practical to implement

## Disadvantages
- Not good for V10
- Constant step size can lead to problems (especially for large steps)

### Example

*Outcome of One "Up and Down" Simulated Test*

![Graph showing the outcomes of a simulated test with two curves: one for no penetration and another for penetration. The graph includes markers for each run number and indicates the true V50 value.]*
Langlie Method

Details of Implementation

START

Choose initial inputs:
1) Lower limit (LL)
2) Upper limit (UL)

V1 = LL + (UL - LL) / 2

Run Trial

Record Velocity and Result

Find next velocity (n+1):
1) Start at the nth trial
2) Find the previous nth trial so that there are an equal number of penetrations and no penetrations between nth and pth trial.

pth trial found

Vn > Vp

Vn+1 = Vp + (Vn - Vp) / 2

Vp > Vn

Vn+1 = Vn + (Vn - Vp) / 2

Vn > Vp

nth trial DID NOT penetrate

Vn+1 = Vn + (Vn - Vp) / 2

nth trial DID penetrate

Vn+1 = LL + (UL - LL) / 2

Background

- Numerous modified versions exist
- Developed in early 60s

Advantages

- Useful for estimating V50
- Has an adaptive step size

Example

Outcome of One Langlie Method Simulated Test

No Penetration

Penetration

True V50

Disadvantages

- Not designed for d-optimal curve fitting
- Not as easy to implement as up and down method
**Details of Implementation**

- If projectile **does** penetrate armor, decrease velocity.

- If projectile **does not** penetrate armor k times in a row, increase velocity.

- The step size is chosen based on the standard deviation of the predicted response curve.

- Targets Pth quantile of interest where

\[
P = 1 - \left( \frac{1}{2} \right)^\frac{1}{k}
\]

- Typically, k=2 (P≈0.3) or k=3 (P≈0.2)

**Background**

- Similar to Up and Down Method

- Not typically used in armor testing

**Advantages**

- Useful for estimating percentiles away from the median

- Easy to implement (similar to Up and Down method)

**Example**

Outcome of One "k-in-a-row" Simulated Test

![Graph showing velocity vs. run number with symbols indicating no penetration and penetration, along with the true V_{50}]

**Disadvantages**

- Less accurate for estimating V50

- A constant step size is susceptible to problems
Robbins Monroe

Details of Implementation
- Start the test at predicted V50.
- Determine the velocity of the next shot using

\[ x_{n+1} = x_n - c(y_n - P)/n \]

where \( c \) is an arbitrary constant, \( y_n \) is the outcome of the nth trial \((0,1)\), \( P \) is the desired percentile of interest and \( n \) is the number of trials. \( C \) is optimal when:

\[ c_{opt} = \left[ F'(V_p) \right]^{-1} \]

where \( F \) is the response curve and \( V_p \) is the velocity at the pth percentile
- Step size decreases as \( n \) increases

Background
- Developed in 1951
- Numerous variants of this method exist
- Used in armor testing by ARL

Advantages
- Useful for estimating all quantiles
- A dynamic step size has advantages

Disadvantages
- Justification for values of \( c \) may seem arbitrary, poor choices of \( c \) can lead to inaccurate results
- Poor guess of the velocity of the first shot can lead to slow convergence and/or convergence to an inaccurate result

Example
Outcome of One Optimal Robbins Monroe Simulated Test

![Graph showing velocity over run number with no penetration and penetration marks, along with true V50 value.](image)
Neyer’s Method

Details of Implementation

- **Phase 1: Generate penetrations and non-penetrations.** Bounds the problem. Determines if initial gate is too far left, right or narrow.

- **Phase 2: Break separation.** Provides unique MLE coefficient estimates and an indication that velocity is in the ballpark of V50.

- **Phase 3: Refine model coefficients.** Use D-optimality criterion to dictate ensuing shots.

Background

- Developed by Neyer in 1989
- First to propose a systemic method for generating a good initial design

Advantages

- Initial design is useful for quickly estimating model coefficients
- Robust to misspecification of input parameters

Disadvantages

- Requires coding and capability to do maximum likelihood estimation

Example

![Graph showing relationship between Run Number and Velocity (ft/s)]
## 3Pod

### Details of Implementation

- **Phase 1: Generate penetrations and non-penetrations.** Similar to rules to Neyer’s method. Uses slightly different logic and different step sizes.

- **Phase 2: Break separation.** Relies more heavily on conditional logic than Neyer’s method.

- **Phase 3: Refine model coefficients (and estimate of Vp).** A portion of resources is devoted to D-optimal algorithm and the other portion is used for placing shots near Vp (velocity percentile value of interest) using Robbins Monroe Joseph method.

### Background

- Developed by Wu in 2013
- Similar to Neyer’s Method

### Advantages

- Similar to Neyer’s Method, good initial design

### Disadvantages

- Requires maximum likelihood estimation
- More complex than Neyer’s method

### Example

![Graph showing data points and run number](image)
Example of 3Pod Results

- Example of 30 Shots for 3-Phase Approach (3Pod)
Simulation Comparison
Simulation Factors and Responses

Response

1. V50 Error
2. V10 Error

Calculated as the difference between the “true” V50 (or V10) and the V50 (or V10) estimated with the simulated runs

Factors

1. Estimator (Probit-MLE, Arithmetic Mean)
2. Method (Up Down Method, 3Pod, Langlie, etc...)
3. Stopping criteria (“3&3”, break separation)
4. $\mu_{\text{guess}} \mid (\mu_{\text{true}} - 2\sigma_{\text{true}}, \mu_{\text{true}}, \mu_{\text{true}} + 2\sigma_{\text{true}})$
5. $\sigma_{\text{guess}} \mid (1/3\sigma_{\text{true}}, 1/2\sigma_{\text{true}}, 2\sigma_{\text{true}}, 3\sigma_{\text{true}})$
Runs for Stopping Criteria

- UD
- LM
- DRM
- 3POD
- NM
- RMJ

\[ \mu_G = \mu_T - 2\sigma_T \]
\[ \mu_G = \mu_T \]
\[ \mu_G = \mu_T + 2\sigma_T \]

Runs for 33SC
Runs for BSSC

\[ \frac{\sigma_G}{\sigma_T} = \frac{1}{3} \]
\[ \frac{\sigma_G}{\sigma_T} = \frac{1}{2} \]
\[ \frac{\sigma_G}{\sigma_T} = 1 \]
\[ \frac{\sigma_G}{\sigma_T} = 2 \]
\[ \frac{\sigma_G}{\sigma_T} = 3 \]

Legend
- Intercept = 10.5
- Units in Runs

Coefficient Estimate
- 2
- 0
- -2
Recommend 3Pod or Neyer Method

Provides entire logistic model curve fit

Robust estimate for V50 and V10

D-optimal approach
2-D Sensitivity Test Designs
Sensitivity Test Designs with Two Factors

- Response is binary
- no interaction terms
- Two continuous factors
- Primary factor is velocity

![Contour of Probability of Penetration](image-url)

- Projectile Impact Velocity vs. Obliquity Angle
- Armor plate, projectile path, obliquity angle, yaw angle
Practical Multi-Factor Sequential Design

Practical multi-factor sequential designs:

1. Brute force use of single factor sequential designs in multi-dimensional design space
   - Intuitive design and easy to implement

<table>
<thead>
<tr>
<th>Obliquity Angle (deg)</th>
<th>S</th>
<th>M</th>
<th>L</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3Pod</td>
<td>3Pod</td>
<td>3Pod</td>
</tr>
<tr>
<td>20</td>
<td>3Pod</td>
<td>3Pod</td>
<td>3Pod</td>
</tr>
<tr>
<td>40</td>
<td>3Pod</td>
<td>3Pod</td>
<td>3Pod</td>
</tr>
</tbody>
</table>

   Each 3Pod uses velocity as factor

2. Propose a modified sequential design to search D-optimal points across multiple factors

3. Bayesian Sequential Design by Dror and Steinberg (2008)
   - Established, practical sequential design for multiple factors
   - Uses prior information about armor performance to search for D optimal points
1. 3Pod, Neyer, and DS focus on D-optimality
   - D-optimality is a widely accepted design criteria
   - D-optimality is a widely accepted design criteria
   - minimizes the confidence ellipsoid on coefficients

### Calculation of D-optimality
The D-optimality designs criterion for fitting a logistic model maximizes the determinant of the information matrix among all competing designs ($\Omega$).

$$\text{Max}_\Omega |I(\beta)|$$

The fisher information matrix is

$$I(\beta) = |X'\Sigma X|$$

$X$ is the $m \times p$ model matrix.

$\Sigma$ is the variance-covariance matrix for the $m \times 1$ vector of binomial variables, each being $\sum_j y_{ij}$, the sum of events at the $i^{th}$ design point.

$\Sigma$ is an $m \times m$ diagonal matrix with the $i^{th}$ diagonal element being $n_i P_i (1 - P_i)$.

2. Multi-factor sequential designs are compared in terms of D-efficiency
   - The D-efficiency of a candidate design is calculate as

$$\text{D-efficiency} = \frac{|X'\Sigma X|_{\text{Candidate Design}}}{|X'\Sigma X|_{\text{D-optimal Design}}}$$
D-Optimal Design with 1 Factor

- The single factor logistic regression model, \( \ln \left( \frac{p}{1-p} \right) = \beta_0 + \beta_1 x_1 \), can be reparametrized in terms of location-scale parameters as
  \( \ln \left( \frac{p}{1-p} \right) = \frac{x_1 - \mu}{\sigma} \), where \( \mu = -\frac{\beta_0}{\beta_1} \) and \( \sigma = \frac{1}{\beta_1} \)
  - \( \mu \) is \( V_{50} \) and \( \sigma \) is the amount of slope in the curve
  - Figure 1 illustrates various logistic model curve fits

- Abdelbasit and Plackett derived the determinant of the fisher information matrix:
  \( |I| = \frac{n^2 w_1 w_2}{\sigma^2} (x_1 - x_2)^2 \), where
  \( w_i = p_i (1 - p_i) \) and \( x_i = \ln \left( \frac{p_i}{1-p_i} \right) \sigma + \mu \), for \( i = 1, 2 \).
  - Assumes a 2 point design where \( p_1 \) is symmetrical to \( p_2 \), and \( n \) is the number of runs at each point.

- Abdelbasit and Plackett showed the solution is the \( \delta \) that maximizes \( |I| \), where \( p_1 = \delta \) and \( p_2 = 1 - \delta \)

- The D-optimal solution (Figure 2) is \( p_1 = 0.176 \) and \( p_2 = 0.824 \)
  - Meaning that half of the shots are fired at \( V_{17.6} \) and the other half are fired at \( V_{82.4} \)

D-Optimal 1-Factor Design
Specifies Shots at \( V_{17.6} \) and \( V_{82.4} \)

Figure 1 – Example Model Fits

Figure 2 – Numerical Solution

D-Optimal Design with 2 Factors

- The dual factor logistic regression model can be expressed as
  \[ \ln \left( \frac{p}{1-p} \right) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 \] or \[ \ln \left( \frac{p}{1-p} \right) = u \]

- Sitter and Torsney (1995), and Jia and Meyers (2001) developed a 4 point D-optimal design
  - 2 points are placed at the lower obliquity angle setting \((\theta_L)\) and 2 points are placed at the upper setting \((\theta_U)\)
  - Results in a location-scale parametrization:
    \[
    \begin{align*}
    \mu_L &= -\frac{\beta_0}{\beta_2} - \frac{\beta_1 \theta_L}{\beta_2}, \\
    \mu_U &= -\frac{\beta_0}{\beta_2} - \frac{\beta_1 \theta_U}{\beta_2}, \\
    \sigma &= \frac{1}{\beta_2}
    \end{align*}
    \]
  - 4 point D-optimal design:

<table>
<thead>
<tr>
<th>Location</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-u - \beta_0, 0))</td>
<td>(w)</td>
</tr>
<tr>
<td>((0, -u - \beta_0))</td>
<td>(w)</td>
</tr>
<tr>
<td>((u - \beta_0, 0))</td>
<td>(\frac{1}{2} - w)</td>
</tr>
<tr>
<td>((0, u - \beta_0))</td>
<td>(\frac{1}{2} - w)</td>
</tr>
</tbody>
</table>

- where \(u\) and \(w\) are numerically solved for using equations:
  \[
  u^2 (3 + 3e^u + 2u - 2ue^u) + \beta_0^2 (1 + e^u + 2u - 2ue^u) + \sqrt{u^4 + 14\beta_0^2 u^2 + \beta_0^4 (1 + e^u + u - ue^u)} = 0
  \]
  \[
  w = \frac{(-u^2 + 6u\beta_0 - \beta_0^2 + \sqrt{u^2 + 14\beta_0 u + \beta_0^2})}{24\beta_0 u}
  \]

- \(\delta = 0.227\)
- \(p_1 = 0.227\)
- \(p_2 = 0.773\)
- \(w = 0.225\)

Figure 3 – Example Model Fit
\[
\mu_L = 1392, \mu_U = 1932, \sigma = 120
\]

Figure 4 – Numerical Solution
\[
\delta = 0.227
\]
\[
p_1 = 0.227 \\
p_2 = 0.773 \\
w = 0.225
\]
Expanding 3Pod’s D-Optimal Search to Two Factors

- Proposed strategy to implement 3Pod in a two factor space
  1. Conduct initial design with velocity as the factor at zero degree obliquity angle
  2. Conduct an additional initial design with velocity as the factor at 45 degree obliquity angle
  3. Select next point by searching velocity settings that maximize the determinant of the Fisher information matrix.
     » Constrain search to velocities at 0 and 45 degree obliquity since we know that is where the 4 point locally d-optimal points is

Execute initial design at 0 deg and 45 deg obliquity angle → Search two-factor design space for candidate point that maximizes $|X'\Sigma X|$ → Execute point at $\star$ → Update model coefficients and search for new point

![Graphs showing impact velocity vs. obliquity angle with contour lines and color scale for determinant of Fisher information matrix.](image)
Theoretical Improvement

- We can calculate the improvement gained by expanding the search to additional factors, since we can analytically solve for the D-optimal design.

- Three 30 run designs considered:

<table>
<thead>
<tr>
<th>Design 1</th>
<th>Design 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Obliquity Angle</strong></td>
<td><strong>Obliquity Angle</strong></td>
</tr>
<tr>
<td>0 deg</td>
<td>45 deg</td>
</tr>
<tr>
<td>15 runs</td>
<td>15 runs</td>
</tr>
<tr>
<td>(7 runs @ V17.6, 5 runs @ V82.4)</td>
<td>(7 runs @ V17.6, 5 runs @ V82.4)</td>
</tr>
<tr>
<td>15 runs</td>
<td>15 runs</td>
</tr>
<tr>
<td>(7 runs @ V22.7, 8 runs @ V77.3)</td>
<td>(7 runs @ V22.7, 8 runs @ V77.3)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>D-optimal Design</th>
<th>Design 1</th>
<th>Design 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><strong>Obliquity Angle</strong></td>
<td><strong>Obliquity Angle</strong></td>
<td><strong>Obliquity Angle</strong></td>
</tr>
<tr>
<td>0 deg</td>
<td>45 deg</td>
<td>0 deg</td>
<td>22.5 deg</td>
</tr>
<tr>
<td>15 runs</td>
<td>15 runs</td>
<td>15 runs</td>
<td>10 runs</td>
</tr>
<tr>
<td>(7 runs @ V17.6, 8 runs @ V77.3)</td>
<td>(7 runs @ V17.6, 8 runs @ V77.3)</td>
<td>(5 runs @ V17.6, 8 runs @ V77.3)</td>
<td>(5 runs @ V17.6, 8 runs @ V77.3)</td>
</tr>
<tr>
<td>15 runs</td>
<td>15 runs</td>
<td>15 runs</td>
<td>10 runs</td>
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<tr>
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<td>(5 runs @ V22.7, 8 runs @ V77.3)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>[(X'\Sigma X)]:</th>
<th>D-efficiency:</th>
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<tbody>
<tr>
<td>D-optimal Design</td>
<td>1.5E9</td>
<td>1.0</td>
</tr>
<tr>
<td>Design 1</td>
<td>1.4E9</td>
<td>.896</td>
</tr>
<tr>
<td>Design 2</td>
<td>1.0E9</td>
<td>.600</td>
</tr>
</tbody>
</table>

- These designs are infeasible in practice because we don’t have prior knowledge of coefficients.
  - We must run simulations that include an initial design to determine practical improvement.
Simulation Setup

12 run factorial experiment
  - Response: D-efficiency
  - Factors:
    - Methods
      - 3Pod w/ 1-factor D-optimal search (3Pod-1D)
      - 3Pod w/ 2-factor D-optimal search (3Pod-2D)
      - Dror-Steinberg Method (D-S)
      - Langlie Method
    - Sample Sizes
      - 60, 120

Method Input parameters
  - D-S requires prior uniform distributions on model coefficients
  - 3Pod requires specification of $\sigma_G$ and $\mu_G$ at 0 and 45 degree obliquity angle
  - To make a fair comparison, inputs for each method need to be equivalent

Constant inputs into simulation
  - Assumed true logit model: $b_T = [b_{0T} \ b_{1T} \ b_{2T}] = [-11.6 \ -1 \ .0083]$
  - Number of simulations per factorial trial: 1,000
Simulation Results

3Pod with 1 Factor D optimal Search
- Runs at 0 deg
- Runs at 45 deg
- True D-optimal point

3Pod with 2 Factor D optimal Search
- Runs at 0 deg
- Runs at 45 deg
- True D-optimal point

Dror-Steinberg
- Runs at 0 deg
- Runs at 45 deg
- True D-optimal point

Langlie Method
- Runs at 0 deg
- Runs at 45 deg
- True D-optimal point
Simulation Results

**60 Runs**
- 3Pod 1D (median=0.66)
- 3Pod 2D (median=0.66)
- D-S (median=0.70)
- Langlie (median=0.64)

**120 Runs**
- 3Pod 1D (median=0.76)
- 3Pod 2D (median=0.81)
- D-S (median=0.82)
- Langlie (median=0.64)
Recommendations

D-S and 3Pod2D perform best

Further investigation into the practicality, and robustness of D-S is needed