CONVERGENCE BEHAVIOR OF SERIES SOLUTIONS OF THE LAMBERT PROBLEM

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Lambert's problem, to find the unique conic trajectory that connects two points in a spherical gravity field in a given time, is represented by a set of transcendental equations due to Lagrange. The associated Lagrange equations for the orbital transfer time can be expressed as series expansions for all cases. Power series solutions have been published that reverse the functionality of the Lagrange equations to provide direct expressions for the unknown semi-major axis as an explicit function of time. The convergence behavior of the series solutions is examined over the range of possible transfer angles and flight times. The effect of arbitrary precision calculations is shown on the generation of the series coefficients.

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Lagrange derived a set of transcendental equations that determine the time of flight on an orbital trajectory as a function of the semi-major axis of the conic section that connects two points in a spherical gravity field. However, in the Lambert problem, the semi-major axis is desired as a function of the given time of flight rather than the reverse, so Lagrange's equations (or other reformulations of them) must be supplemented by some sort of root-finding technique to form a complete solution. Alternatively, power series have been published (Thorne 2004) that algebraically reverse the functionality of the Lagrange equations to provide a direct solution for the unknown semi-major axis as an explicit function of the given time of flight.

The article examined the convergence behavior of the series solutions to the Lambert problem over the range of possible transfer angles and flight times that approach the full period of the orbit. Also, the effect of arbitrary precision arithmetic was shown for the calculation of the series coefficients.

BACKGROUND – SERIES SOLUTIONS OF THE LAMBERT PROBLEM

For convenience, the series solutions of the Lambert problem (Thorne 2004) are repeated here for the discussion of convergence properties. Lambert's theorem states that the orbital transfer time (t) between two known positions in the 2-body orbital problem is dependent only on the semi-major axis (a), given two fixed position vectors and a known gravitational constant. Lagrange proved this theorem and derived elegant equations that show this functional dependence (Battin 1987, 287). Apart from the limiting cases of straight-line and parabolic transfers, the possible orbital paths fall into three categories: Hyperbolic, A-type elliptical, or B-type elliptical. Hyperbolic transfers take less time than the unique parabolic solution, A-type elliptical transfers take longer than the parabolic time but less than minimum-energy time, and B-type elliptical transfers take longer than minimum-energy time. Also, a leading number 1 or 2 will indicate whether the transfer angle is less than 1 or greater than 2 π radians. Thus, the possible transfer types (1H, 2H; 1A, 2A; and 1B, 2B) are apart from the exact parabolic and minimum-energy cases that can be calculated from the given inputs to the Lambert problem. Typically, A-type elliptical transfers involve orbital arcs that connect position vectors going the "short way" around the ellipse, while B-type transfers generally go the "long way" around the ellipse. Although this language is commonly used to describe elliptical orbit arcs, care should be used to select the correct version of the Lagrange equations based on time and transfer angle as explained above. The basic problem geometry is shown in Figure 1.

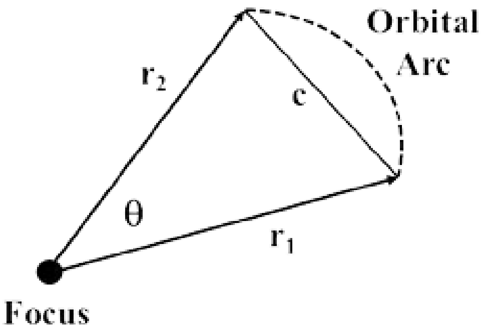

Figure 1. Lambert Problem Geometry

If the transfer angle is less or more than π radians, there is a corresponding sign change (±) in the Lagrange equation, as shown in Eq. 1:

$$\text{1A, 2A:}\ t = \frac{\sqrt{a^3}}{k}\{(\alpha - \sin\alpha) \mp (\beta - \sin\beta)\}$$

Eq.1

where

$$\alpha = 2\sin^{-1}\sqrt{\frac{s}{2a}}, \beta = 2\sin^{-1}\sqrt{\frac{s-c}{2a}}$$

Eq. 2

In Eq. 2, c is the chord, which is a line segment connecting the two observed position vectors, and s is one-half the sum of the lengths of the position vectors and the chord, $s = (r_1 + r_2 + c)/2$ as shown in Figure 1. The gravitational constant is k. These quantities are known from the given observations and are not dependent on the transfer time, t_p. The unique parabolic flight time for a given geometry, t_p, is given by Eq. 3, and the unique minimum-energy flight time can be found from Eqs. 1 and 2 by letting $a = s/2$.

$$t_p = \frac{\sqrt{2}}{3}\frac{s^{3/2}}{k}\left((1 \mp \left(\frac{s-c}{s}\right)^{3/2}\right)$$

Eq.3

Many excellent iterative techniques exist to solve the Lagrange equations. (Thorne and Bain 1995). However, since a is the unknown quantity to be found, it would be quite useful to find a direct solution to avoid the need for any type of root-finding technique. Series reversion may be used to solve Eq. 2 as follows:

$$a = \left(\frac{s}{2}\right) \sum_{n=0}^{\infty} \frac{B_n}{[\mathrm{H}, \mathrm{A}]} \left(\frac{t}{t_p} - 1\right)^{n-1}$$

Eq.4

However, series reversion may result in a small radius of convergence as well as numerical errors in the calculation of high-order terms. Next, we address the effect of machine precision on the evaluation of Eq. 4.

EFFECT OF MACHINE PRECISION ON THE HYPERBOLIC AND SHORT-WAY ELLIPTIC SERIES

The series solution for the hyperbolic and short-way elliptic cases, Eq. 4, originally appeared to be asymptotically convergent (Vallado 2001, 476–485). As described by Dr. Richard H. Battin in conversations with the author in 1990, asymptotic convergence occurs when a divergent series can produce useful results at limited orders because the initial behavior is convergent prior to later growth of the series terms, which was the case for Eq. 4. However, based on recent numerical experiments using arbitrary precision, the series appears to behave in a convergent manner because the growth in higher order terms was apparently a result of machine precision limitations based on the technology of the time and not due to asymptotic convergence.

As a numerical example of the Lambert problem using earth-based physical units, consider an object in earth orbit with two known position observations and a transfer time. Suppose that the magnitudes of the position vectors are r_1 = 8000 km and r_2 = 8010 km, the transfer angle is 140°, and the earth's gravitational constant is 398600.44144982 km^3/s^2. If the observed time of flight of the orbital transfer is exactly 2550 s, the goal is to determine the semi-major axis of the unknown orbit. This is an example of a short-way elliptical transfer, which can be solved using Eq. 4 with arbitrary precision calculations to demonstrate the improvement over finite precision calculations.

Table 1 shows the cumulative effect of numerical precision of using finite precision (FP) calculations vs. arbitrary precision (AP) calculations to determine the values of the series coefficients, series terms, and partial sums. In this case, finite precision calculations maintained 17 digits based on an extended real variable declaration in the original Pascal programming language. Using arbitrary precision with a symbolic manipulation program such as Mathematica (MMA) effectively removes the limit on the precision level of the calculations by setting it to any level desired.

The first column in Table 1 shows the index in the series, the second and third show the series coefficients, the fourth and fifth show the value of the full series term with the time argument, and the sixth and seventh show the partial sums that are the calculated value of semi-major axis in kilometers. The series coefficients, B_n, show good agreement through an index value of 23. Under finite precision, the B_n begin to diverge above an index value of 23, whereas they continue to converge indefinitely

Table 1. Example History of Series Behavior – Comparison of Finite Precision (FP) vs. Arbitrary Precision (AP) Results to Order 35

Index	FP b[n]	AP b[n]	FP term[n]	AP term[n]	FP sum (km)	AP sum (km)
1	2341.5257009	2341.5257009	3021.9126260	3021.9126260	3021.9126260	3021.9126260
2	4159.7690280	4159.7690280	4159.7690280	4159.7690280	7181.6816540	7181.6816540
3	1549.5393497	1549.5393497	1200.6588744	1200.6588744	8382.3405283	8382.3405283
4	-167.2452322	-167.2452322	-100.4125041	-100.4125041	8281.9280242	8281.9280242
5	51.1296198	51.1296198	23.7861219	23.7861219	8305.7141462	8305.7141462
6	-21.2401119	-21.2401119	-7.6564051	-7.6564051	8298.0577411	8298.0577411
7	10.3514416	10.3514416	2.8912521	2.8912521	8300.9489932	8300.9489932
8	-5.6117878	-5.6117878	-1.2145164	-1.2145164	8299.7344768	8299.7344768
9	3.3174596	3.3174596	0.5563202	0.5563202	8300.2907970	8300.2907970
10	-2.1288137	-2.1288137	-0.2766138	-0.2766138	8300.0141832	8300.0141832
11	1.4838952	1.4838952	0.1494020	0.1494020	8300.1635852	8300.1635852
12	-1.1208531	-1.1208531	-0.0874418	-0.0874418	8300.0761434	8300.0761434
13	0.9078658	0.9078658	0.0548793	0.0548793	8300.1310228	8300.1310228
14	-0.7747516	-0.7747516	-0.0362883	-0.0362883	8300.0947344	8300.0947344
15	0.6827783	0.6827783	0.0247800	0.0247800	8300.1195144	8300.1195144
16	-0.6102833	-0.6102834	-0.0171621	-0.0171621	8300.1023524	8300.1023524
17	0.5453631	0.5453631	0.0118834	0.0118834	8300.1142358	8300.1142358
18	-0.4819197	-0.4819210	-0.0081367	-0.0081367	8300.1060991	8300.1060991
19	0.4174042	0.4174044	0.0054607	0.0054607	8300.1115598	8300.1115597
20	-0.3514535	-0.3514281	-0.0035627	-0.0035624	8300.1079971	8300.1079973
21	0.2850735	0.2848903	0.0022391	0.0022377	8300.1102362	8300.1102350
22	-0.2205273	-0.2193769	-0.0013422	-0.0013352	8300.1088941	8300.1088999
23	0.1621976	0.1567467	0.0007649	0.0007392	8300.1096590	8300.1096391
24	-0.1279172	-0.0988385	-0.0004674	-0.0003612	8300.1091916	8300.1092779
25	0.1347753	0.0472689	0.0003816	0.0001338	8300.1095732	8300.1094117
26	-0.3354922	-0.0032989	-0.0007360	-0.0000072	8300.1088371	8300.1094045
27	0.6277174	-0.0322417	0.0010671	-0.0000548	8300.1099042	8300.1093497
28	-0.8371108	0.0589831	-0.0011026	0.0000777	8300.1088016	8300.1094274
29	4.1027220	-0.0770060	0.0041873	-0.0000786	8300.1129889	8300.1093488
30	-37.2049232	0.0868005	-0.0294223	0.0000686	8300.0835665	8300.1094174
31	305.7985549	-0.0892027	0.1873826	-0.0000547	8300.2709491	8300.1093628
32	-2017.8173395	0.0853155	-0.9580598	0.0000405	8299.3128893	8300.1094033
33	8919.4675674	-0.0764208	3.2814571	-0.0000281	8302.5943464	8300.1093752
34		0.0638904		0.0000182		8300.1093934
35		-0.0490998		-0.0000108		8300.1093825

using arbitrary precision. At index value 35, the last partial sum using arbitrary precision results in a 8300.1093825 km for semi-major axis, which is correct to less than 1 meter. Substituting this value for the semi-major axis in the original Lagrange time-of-flight Eq. 1 as a check, the resulting transfer time is 2550.000002 s. The data from Table 1 can also be graphed to show the beneficial effect of arbitrary machine precision, as seen in Figure 2.

In Figure 2, the magnitude of the series terms is plotted against the index value. The first curve, denoted “Pascal,” was generated with a finite precision Pascal compiler, and the numerical results were included in Table 1 based on the previous discussion. The second curve, denoted “MMA” was generated in Mathematica with no limit on arithmetic precision, and those data were also included in Table 1. The two curves are shown with the same vertical scale to emphasize the dramatic difference in their behavior, and very significantly to show the important result that the power series solution is convergent rather than divergent, as had been previously assumed based on earlier numerical research using finite precision arithmetic. In short, this result confirms the numerical utility of the power series solution for the Lambert Problem of initial orbit and trajectory determination.

The magnitude of the coefficients continues to decrease uniformly out to 150 terms as shown in Figure 2 and to more than 300 terms based on additional numerical experiments, making the series solution much more useful at high order. As the index value gets near to the number digits of machine

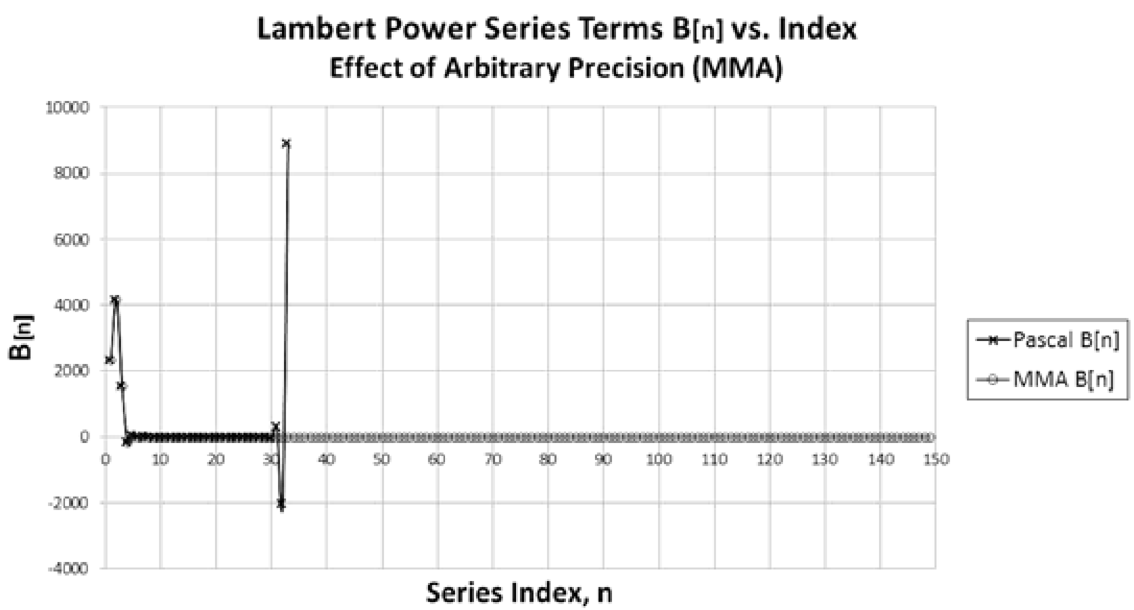

Figure 2. Effect of Arbitrary Precision on Convergence

precision, the calculations can lose significance because the algorithm produces small differences between very large numbers. Arbitrary precision arithmetic addresses this problem, so there is no loss in precision during the arithmetic calculations.

In Figure 3, the series coefficients are shown at a much smaller scale than in Table 1 in order to see the convergence behavior in detail. It can be seen that the series coefficients have alternating signs and exhibit a beat phenomenon in the sinusoidal decay of their magnitudes.

expansion about the region where the semi-major axis approaches infinity on a parabolic transfer, so the accuracy of the series solution is best near the parabolic time. The second series solution to the Lagrange equations uses an expansion about the point where the semi-major axis reaches its minimum positive value, which corresponds physically to a minimum-energy transfer arc. This series solution provides a reasonably accurate solution for the range from the minimum-energy transfer time up to a transfer time that is approximately 1.5 times the minimum-energy transfer time, based on numerical investigation. The

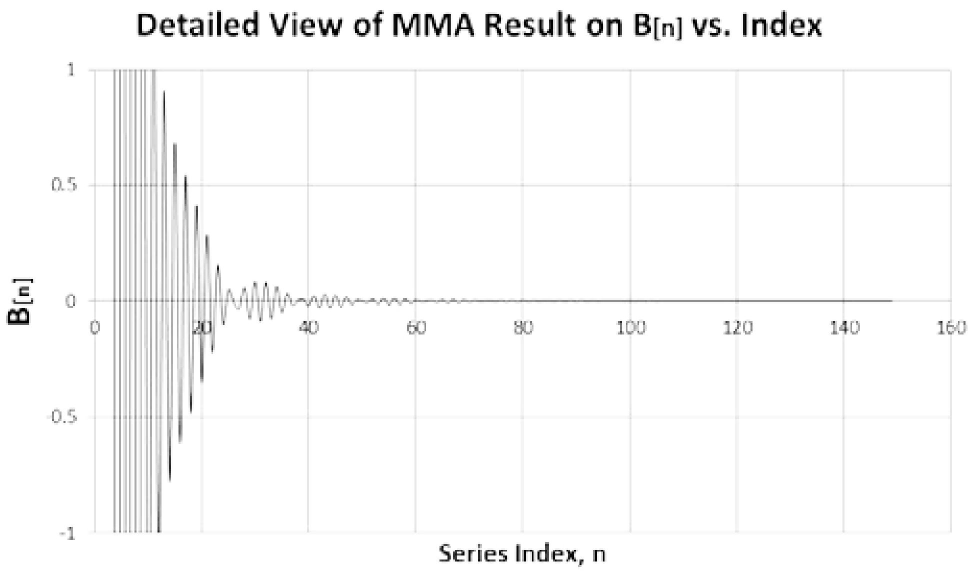

Figure 3. Detail of Series Convergence Behavior

CONCLUSIONS

The series solutions for the Lambert problem show good convergence properties near their expansion points, as would be expected. The first series solution to the Lagrange equations uses an third series solution to the Lagrange equations uses an expansion about the region where the semi-major axis approaches infinity as the transfer time also approaches infinity. Physically, this means that the transfer time approaches the period of the closed orbit.

For the series solution for the hyperbolic and short-way elliptic cases, a combination of series reversion and inversion results in better convergence properties than reversion alone. However, this series will appear to be asymptotically convergent if the coefficients are calculated using finite precision arithmetic. Based on numerical investigation, this divergent behavior completely disappears out to 300 terms when using arbitrary precision calculations, which would suggest that the series solution is actually convergent.

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